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SHORTER COMMUNICATIONS

ONSET OF CONVECTION IN A POROUS LAYER SATURATED BY AN IDEAL GAS

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NOMENCLATURE

C_p,	specific heat at constant pressure;
$\mathbf{g}$,	gravitational acceleration;
H,	layer depth;
k,	thermal conductivity of medium;
K,	permeability;
M,	molecular weight of gas;
p,	pressure;
R,	Rayleigh number, defined by equation (14);
R^*,	universal gas constant;
t,	time;
T,	temperature;
u, v, w,	velocity components;
x, y, z,	cartesian coordinates.

Greek symbols

α,	thermal expansivity;
β,	isothermal compressibility;
Γ,	applied temperature gradient;
ε,	porosity;
μ,	viscosity of fluid;
ρ,	density of fluid.

Subscripts

A,	adiabatic value;
f,	refers to fluid;
m,	refers to medium;
0,	steady state (hydrostatic distribution);
s,	value at surface $z = 0$.

Superscript

$'$,	perturbation variable.

INTRODUCTION

MOST theoretical work on the onset of convection in a porous layer heated below has been done using the Boussinesq–Oberbeck approximation, according to which density differences are considered in the buoyancy term but otherwise the fluid properties are taken to be constant, and as a result the fluid is taken as being quasi-incompressible. A noteworthy exception is the paper by Straus and Schubert [1], who derived non-Boussinesq equations for a general fluid before concentrating on convection of water in a geothermal context. It appears that Saatdjian [2] is the first to explicitly consider a porous layer saturated with an ideal gas.

The particular problem considered in ref. [2] is the onset of convection in a medium bounded by two horizontal isothermal impermeable planes, a problem which is analogous to the Rayleigh–Bénard problem for a viscous fluid. For this Rayleigh–Darcy problem it is well known that the criterion for convection to occur with a Boussinesq fluid is that the Rayleigh number R exceeds the value $4\pi^2$. Saatdjian calculated critical values for R with an ideal gas. He found that they depended on his quantity $R_T = T_p/T_{\text{amb}}$ (where T_{amb} is the ambient temperature (300 K) and T_p is related to the average temperature in the layer $\bar{T}$ by $\bar{T} = T_p + T_{\text{amb}}/2)$ and ranged from a value larger than $4\pi^2$ to values smaller than $4\pi^2$. These results would suggest that compressibility could provide either a stabilizing or destabilizing effect according to the value of T_p. This two-sided effect is surprising. Saatdjian explained the value higher than $4\pi^2$ as "due to the fact that the mixed mean temperature is higher than the average temperature once convective movement appears. Thus, the real Ra_c numbers are a bit below our tabulated values". (A referee of the present paper noted the possibility that approximation of the reference density distribution by a first order polynomial could account for the small numerical discrepancy, if any.) That may be so, but the author suggests that the two-sided effect is probably an artifact of the scaling used by Saatdjian, who introduced T_{amb}, l and P_0 as temperature, length and pressure reference parameters, respectively. Here, T_{amb} is a fixed temperature (300 K), l is the thickness of the porous layer and P_0 is taken as $\rho g l$. However, in order for Saatdjian's equations (6) and (7) (with obvious minor misprints corrected) to follow from equations (2) and (4) it is necessary to assume that $T_{\text{amb}} = MP_0/R\rho_0$ where M is the molecular weight of the fluid and R is the universal gas constant. We conclude that his equations are not properly determined, and as a consequence his results are invalid as they stand at present.

It would not be difficult to amend his paper in this respect, but it suffers a more important defect in its failure to include in the energy equation [Saatdjian's equation (3)] the contribution of the work done by the pressure during changes in volume. As we shall see below, this contribution is of prime importance.

The spadework for a correct theory has already been done by Straus and Schubert, but since an ideal gas differs from water in some important respects it is worthwhile to sketch the reformulation of the theory for the case of the gas.

GENERAL THEORY

The governing equations expressing conservation of mass, momentum and energy may be written (essentially as in [1])

$$\varepsilon(\partial\rho/\partial t) + \nabla\cdot(\rho\mathbf{u}) = 0, \tag{1}$$

$$-\nabla p - (\mu/K)\mathbf{u} + \rho\mathbf{g} = 0, \tag{2}$$

$$(\rho C_p)_m \frac{\partial T}{\partial t} + (\rho C_p)_f \mathbf{u}\cdot\nabla T - \frac{T}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p \left(\frac{\partial p}{\partial t} + \mathbf{u}\cdot\nabla p\right) = k\nabla^2 T. \tag{3}$$

The equation of state for an ideal gas is

$$p = \rho R^* T/M. \tag{4}$$

Equation (2) expresses Darcy's law, and is valid for sufficiently small seepage velocity $\mathbf{u}$. In equation (3) viscous dissipation has been neglected. It is convenient to introduce the thermal expansivity α and isothermal compressibility β defined by

$$\alpha = (-1/\rho)[(\partial\rho/\partial T)_p] \quad \text{and} \quad \beta = (1/\rho)[\partial\rho/\partial p)_T]. \quad (5)$$

For an ideal gas we have $\alpha = 1/T$ and $\beta = 1/p$.

We take cartesian coordinates (x, y, z) with the z-axis in the direction of $\mathbf{g}$. We suppose that the upper surface (at $z = 0$) is maintained at temperature T_s and the lower surface (at $z = H$) is maintained at temperature $T_s + \Gamma H$. Thus Γ denotes the imposed temperature gradient. We assume that k is constant. The steady state solution (denoted by subscript zero) is then given by the hydrostatic equations

$$\mathbf{u}_0 = 0, \quad T_0 = T_s + \Gamma z, \quad \mathrm{d}p_0/\mathrm{d}z = \rho_0 g, \quad (6)$$

and

$$\frac{\mathrm{d}\rho_0}{\mathrm{d}z} = \beta_0\rho_0\left(\frac{\mathrm{d}p_0}{\mathrm{d}z}\right) - \alpha_0\rho_0\left(\frac{\mathrm{d}T_0}{\mathrm{d}z}\right) = \beta_0\rho_0^2 g - \alpha_0\rho_0\Gamma. \quad (7)$$

We consider non-oscillatory 2-dim. small disturbances to the static solution. We denote the disturbed variables by

$$\mathbf{u} = (u, 0, w), \; \rho' = \rho - \rho_0, \; T' = T - T_0, \; p' = p - p_0. \quad (8)$$

These are related by the equations

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} + w(\beta_0\rho_0 g - \alpha_0\Gamma) = 0, \quad (9)$$

$$\rho_0 w(C_{p_0}\Gamma - \alpha_0 T g) = k\left(\frac{\partial^2 T'}{\partial x^2} + \frac{\partial^2 T'}{\partial z^2}\right), \quad (10)$$

$$\frac{\partial p'}{\partial x} + \frac{\mu_0 u}{K} = 0, \quad (11)$$

$$\frac{\partial p'}{\partial z} + \frac{\mu_0 w}{K} = \rho' g, \quad (12)$$

where

$$\rho' = \beta_0\rho_0 p' - \alpha_0\rho_0 T', \quad \alpha_0 = \frac{1}{T_0}, \quad \beta_0 = \frac{1}{p_0}.$$

The boundary conditions are

$$w = 0, \quad T' = 0 \quad \text{at } z = 0 \text{ and } z = H. \quad (13)$$

The standard procedure is now to put equations (9)–(13) in non-dimensional form, separate the variables by supposing that all the perturbation variables are periodic in x with some wavenumber, and then eliminate the pressure and the x-component of the velocity. One then has a fourth-order differential equation and four boundary conditions. The eigenvalues of this system can then be found using standard methods. (See [1] and the references contained therein.) The Rayleigh–Darcy number R defined by

$$R = \frac{g\alpha\Gamma\rho^2 C_p K H^2}{\mu k} \quad (14)$$

(with quantities evaluated at some suitable reference temperature such as the surface temperature T_s) is then minimized as a function of the wavenumber. The minimum value of R is the critical value R_c which must be exceeded for convection to occur.

In the Boussinesq–Oberbeck approximation the term $-\alpha_0\rho_0 T'$ for ρ' is retained, but otherwise α_0 and β_0 are taken as zero and ρ_0, C_{p_0}, T_0, μ_0 are regarded as constants. One then finds that $R_c = 4\pi^2$. As a second approximation one can retain the term $-\alpha_0 T_0 g$ in equation (10), the LHS of which can then be written as $\rho_0 C_{p_0}(\Gamma - \Gamma_A)w$, where Γ_A is the adiabatic gradient g/C_{p_0}. If one defines R_A analogously to R but with Γ replaced by Γ_A, then the critical Rayleigh number R_c is given by

$$R_c = 4\pi^2 + R_A.$$

Thus on this approximation the effect of compressibility is stabilizing. An analogous result for the Rayleigh–Bénard problem was first derived by Jeffreys [3].

In the general case, with all the terms in equations (9)–(12) retained, a detailed calculation is required, and we may refer to ref. [1]. For the case of an ideal gas the criterion for convection depends on $\Delta T/T$ and $\beta\rho g H$ (evaluated at the surface) as well as on R and R_A. When some appropriate experimental data is available it will be worthwhile to make the calculation, but in the meantime we can obtain qualitative predictions without recourse to a computer.

A major difference for convection in a porous medium saturated with an ideal gas, from that with water, results from the fact that whereas α increases with increase of T for the case of water, the opposite occurs for the case of a gas. The dominant effect of a decrease of α is to produce a stabilizing effect, since the buoyancy force driving the convection is directly proportional to α. Again, for a gas, the adiabatic temperature gradient increases slightly with T (since C_p, decreases), and this produces a stabilizing effect. Finally, the viscosity of a gas increases with T, leading to a further stabilizing effect. We conclude that, for the case of an ideal gas, all the important non-Boussinesq effects tend to be stabilizing in comparison with the situation for a Boussinesq fluid with properties measured at the temperature of the cooler boundary.

CONCLUSION

We have corrected the theory presented by Saatdjian [2], and have applied the theory formulated by Straus and Schubert [1] to obtain a criterion for the onset of convection in a horizontal layer of a porous medium saturated by an ideal gas. The main effect of compressibility is that the difference $\Gamma - \Gamma_A$, between the applied temperature gradient Γ and the adiabatic temperature gradient Γ_A, replaces Γ in the usual criterion in which compressibility is ignored. Other non-Boussinesq effects cause the gas-filled medium to be more stable than would be predicted if the usual Boussinesq approximation was made and fluid properties at the temperature of the cooler boundary were taken in calculating the Rayleigh number.

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